An $O(\log^4(1/\varepsilon))$ refinement of Tao's construction of commutators close to the identity

Boris Bilich

The result was obtained autonomously by GPT-5.6 Sol High, which thought for only 13 minutes.

The paper is also written by it.

Abstract

Let H be an infinite-dimensional complex Hilbert space. Tao proved that for $0 < \varepsilon \le 1/2$ there are $D, X \in B(H)$ such that $\|[D,X] - \mathbf{1}\| \le \varepsilon$ and $\|D\|\,\|X\| = O(\log^5(1/\varepsilon))$. We improve the exponent from five to four:

$$\|D\|\,\|X\| = O\left(\log^4 \frac{1}{\varepsilon}\right).$$

The construction is Tao's, but the right inverse in that construction is estimated more precisely on scalar point sources. Although its global norm is $O(n^2)$, the Green vector generated by a scalar point source has norm $O(n)$. For Tao's boundary forcing this improves $O(n^3)$ to $O(n^2)$. The point-source estimate is proved by iterating the block isomorphism associated with two Cuntz isometries, interpreting the resulting matrix coefficients as a killed walk, and comparing them with binomial incidence matrices. Combining the resulting $m^{-1/2}$ diffusive decay with Tao's spectral decay gives the required Green bound. All implicit constants are absolute.

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1 Introduction

Throughout the paper, H is an infinite-dimensional complex Hilbert space, $\mathcal{B} = B(H)$, and $\mathbf{1} = \mathbf{1}_{\mathcal{B}}$. All logarithms are natural. We use the operator norm on $\mathcal{B}$ and on matrix algebras over $\mathcal{B}$.

The theorem of Wintner and Wielandt asserts that the identity in a unital Banach algebra cannot be a commutator of two bounded elements [6, 5]. For $B(H)$, however, Brown and Pearcy's description of commutators implies that the identity can be approximated arbitrarily well in norm by bounded commutators [1]. The quantitative problem is to determine the price, in the norms of the two factors, of an approximation with prescribed error.

For $0 < \varepsilon < 1$, set

$$m_H(\varepsilon) = \inf\left\{\|D\|\,\|X\| : D, X \in B(H),\ \|[D,X] - \mathbf{1}\| \le \varepsilon\right\}. \tag{1}$$

Popa observed the lower bound

$$m_H(\varepsilon) \ge \frac{1}{2}\log\frac{1}{\varepsilon}; \tag{2}$$

see [3], and also [4] for a short proof. Starting from the Brown–Pearcy construction, Tao proved

$$m_H(\varepsilon) = O\left(\log^5 \frac{1}{\varepsilon}\right) \qquad (0 < \varepsilon \le 1/2) \tag{3}$$

[4]. Krishna and Johnson subsequently extended Tao's logarithmic estimate, with the same exponent five, to a class of unital C^*-algebras that includes $B(H)$ and the Cuntz algebra $\mathcal{O}_2$ [2]. To the author's knowledge, no smaller logarithmic exponent has previously appeared in this problem.

Our main result is the following.

Theorem 1.1. *There is an absolute constant $C > 0$ such that, for every infinite-dimensional complex Hilbert space H and every $0 < \varepsilon \le 1/2$, there exist $D, X \in B(H)$ satisfying*

$$\|[D, X] - \mathbf{1}\| \le \varepsilon \qquad \text{and} \qquad \|D\|\,\|X\| \le C\left(\log \frac{1}{\varepsilon}\right)^4.$$

Equivalently,

$$\frac{1}{2}\log\frac{1}{\varepsilon} \le m_H(\varepsilon) \le C\left(\log\frac{1}{\varepsilon}\right)^4.$$

The construction itself is unchanged from [4]. For an integer n one solves a nonlinear system

$$Tb = a + \delta F(b) + \delta G(b, b). \tag{4}$$

Here $T : \mathcal{B}^n \to \mathcal{B}^{n-1}$ has a right inverse R satisfying $\|R\| = O(n^2)$, the forcing a is a scalar point source of size n at the last coordinate, and $\|F\| = O(n)$, $\|G\| = O(1)$. The global estimate gives only

$$\|Ra\| \le \|R\|\,\|a\| = O(n^3),$$

which forces $\delta \asymp n^{-5}$ in Tao's fixed-point argument. The new input is the point-source Green estimate

$$\left\|(I - E)^{-1}\eta^{(j)}\right\| = O(n) \qquad (2 \le j \le n), \tag{5}$$

where $R = L(I - E)^{-1}$ and $\eta^{(j)}$ is the scalar unit source at coordinate j. Applied to $a = n\eta^{(n)}$, this yields $\|Ra\| = O(n^2)$ and permits $\delta \asymp n^{-4}$.

The distinction between a worst-case resolvent estimate and a point-source Green estimate is the structural content of the improvement. After iterating the Cuntz block isomorphism, the matrix coefficients generated by $\eta^{(j)}$ become scalar: they count pairs of binary paths whose difference walk remains in the interval $\{2, \ldots, n\}$ and terminates at j. Dropping the killing constraint produces a binomial incidence matrix with norm $O(m^{-1/2})$. Tao's weighted norm supplies an additional $\exp(-cm/n^2)$ factor. Summing in m gives (5).

The exponent four is not asserted to be intrinsic to Tao's parametrization. It is the exponent obtained from the present combination of

$$\|R\| = O(n^2), \qquad \|F\| = O(n), \qquad \|Ra\| = O(n^2).$$

A further improvement would require, for example, a better estimate for R on the range of the nonlinear perturbation, cancellation in the distinguished solution, or a parametrization with a smaller quadratic term. Proposition 4.1 is formulated for every scalar point source, rather than only the endpoint source needed for Theorem 1.1, so that the new estimate is separated from the particular forcing in Tao's construction.

The paper is organized as follows. Section 2 records the commutator template and the diagonal similarity. Section 3 recalls Tao's global right inverse. Section 4 proves the point-source Green estimate. Section 5 solves the nonlinear system at scale $\delta \asymp n^{-4}$, and Section 6 completes the proof of Theorem 1.1.

2 The commutator template

We begin by recording the algebraic part of Tao's construction in the form needed later.

Lemma 2.1 (Commutator computation)**.** *Let $n \geq 2$, let $u, v, b_1, \ldots, b_n \in \mathcal{B}$, and let $\delta > 0$. Define $X = (X_{ij})$ and $D = (D_{ij})$ in $M_n(\mathcal{B})$ by*

$$\begin{aligned} X_{ij} &= \mathbf{1}\,\mathbf{1}_{\{i=j+1\}} + \delta b_i\,\mathbf{1}_{\{j=n\}}, \\ D_{ij} &= \delta^{-1} u\,\mathbf{1}_{\{i=j+1\}} + \delta^{-1} v\,\mathbf{1}_{\{i=j\}} + i\mathbf{1}\,\mathbf{1}_{\{j=i+1\}} + b_i u\,\mathbf{1}_{\{j=n\}}. \end{aligned}$$

With the conventions $b_0 = b_{n+1} = 0$, one has

$$[D, X]_{ij} = \mathbf{1}_{\{i=j\}}\mathbf{1} + \Big([v, b_i] + [u, b_{i-1}] + i\delta b_{i+1} + \delta b_i[u, b_n] - n\mathbf{1}\mathbf{1}_{\{i=n\}}\Big)\mathbf{1}_{\{j=n\}}. \tag{6}$$

Proof. It is enough to prove the assertion for $\delta = 1$: the general identity is obtained by replacing

$$(u, v, b_1, \ldots, b_n) \quad \text{by} \quad (\delta^{-1}u, \delta^{-1}v, \delta b_1, \ldots, \delta b_n)$$

in the $\delta = 1$ identity.

Assume therefore that $\delta = 1$. Let $\operatorname{diag}(v)$ be the diagonal matrix with all diagonal entries equal to v, and let N be the weighted upper shift

$$N_{i,i+1} = i\mathbf{1} \quad (1 \leq i \leq n-1), \qquad N_{ij} = 0 \quad \text{otherwise}.$$

Then

$$D = \operatorname{diag}(v) + X\operatorname{diag}(u) + N.$$

The commutator $[\operatorname{diag}(v), X]$ vanishes off the last column, and its entry in row i of that column is $[v, b_i]$. Since

$$[X\operatorname{diag}(u), X] = X[\operatorname{diag}(u), X],$$

this second commutator also vanishes off the last column, where its row-i entry is

$$[u, b_{i-1}] + b_i[u, b_n].$$

Finally, direct multiplication gives

$$[N, X]_{ii} = \mathbf{1} \quad (1 \leq i \leq n-1), \qquad [N, X]_{nn} = (1-n)\mathbf{1},$$

and the additional last-column entry in row $i < n$ is ib_{i+1}. Adding these three commutators gives (6) for $\delta = 1$, and hence for arbitrary $\delta > 0$. □

Remark 2.2 (Scaling convention). *The last-column coefficient in D is $b_i u$, with no factor of δ. Indeed, the δ-dependent formula is obtained from the $\delta = 1$ construction by the substitution $u \mapsto \delta^{-1}u$, $v \mapsto \delta^{-1}v$, and $b_i \mapsto \delta b_i$; the product in the last column is then $(\delta b_i)(\delta^{-1}u) = b_i u$. This convention is also forced by the direct commutator computation in Lemma 2.1.*

Corollary 2.3 (Diagonal similarity). *Let $u, v, b_1, \ldots, b_n \in \mathcal{B}$ and $\delta > 0$. Suppose*

$$[v, b_i] + [u, b_{i-1}] + i\delta b_{i+1} + \delta b_i[u, b_n] = 0, \qquad 2 \le i \le n-1, \tag{7}$$

and

$$[v, b_n] + [u, b_{n-1}] + \delta b_n[u, b_n] = n\mathbf{1}. \tag{8}$$

Then, for every $\mu > 0$, there exist $D_\mu, X_\mu \in M_n(\mathcal{B})$ such that

$$\|D_\mu\| \le \frac{\|u\|}{\mu^2\delta} + \frac{\|v\|}{\mu\delta} + (n-1) + \sum_{i=1}^{n} \mu^{n-i-1} \|b_i\| \, \|u\| \,, \tag{9}$$

$$\|X_\mu\| \le 1 + \delta \sum_{i=1}^{n} \mu^{n-i+1} \|b_i\| \,, \tag{10}$$

$$\left\| [D_\mu, X_\mu] - \mathbf{1}_{M_n(\mathcal{B})} \right\| \le \mu^{n-1} \left\| [v, b_1] + \delta b_2 + \delta b_1[u, b_n] \right\| . \tag{11}$$

Proof. Let D, X be as in Lemma 2.1, and put

$$S_\mu = \operatorname{diag}(\mu^{n-1}\mathbf{1}, \mu^{n-2}\mathbf{1}, \ldots, \mathbf{1}), \qquad D_\mu = \mu^{-1} S_\mu D S_\mu^{-1}, \qquad X_\mu = \mu S_\mu X S_\mu^{-1}.$$

By (7) and (8), all entries of $[D, X] - \mathbf{1}$ vanish except possibly the $(1, n)$ entry, which is

$$[v, b_1] + \delta b_2 + \delta b_1[u, b_n].$$

Since conjugation by S_μ multiplies the $(1, n)$ matrix unit by μ^{n-1}, this proves (11).

For completeness, the nonzero parts of D_μ are: a lower diagonal with coefficient $(\mu^2\delta)^{-1}u$, a diagonal with coefficient $(\mu\delta)^{-1}v$, the weighted upper shift with weights $1, \ldots, n-1$, and a last column whose row-i entry is $\mu^{n-i-1} b_i u$. The triangle inequality gives (9). Similarly, X_μ is the unweighted lower shift plus a last column with row-i entry $\delta\mu^{n-i+1} b_i$, which gives (10). $\square$

3 The right inverse and its global bound

Choose isometries $u, v \in \mathcal{B}$ with orthogonal ranges whose range projections sum to the identity:

$$u^*u = v^*v = \mathbf{1}, \qquad u^*v = v^*u = 0, \qquad uu^* + vv^* = \mathbf{1}. \tag{12}$$

Such isometries exist because $H \cong H \oplus H$.

For $n \ge 3$, equip

$$\mathcal{X}_n = \mathcal{B}^n, \qquad \mathcal{Y}_n = \mathcal{B}^{n-1}$$

with their supremum norms. We index elements of $\mathcal{Y}_n$ by $i = 2, \ldots, n$. Define

$$(Tb)_i = [v, b_i] + [u, b_{i-1}], \qquad 2 \le i \le n. \tag{13}$$

For $x \in \mathcal{Y}_n$, set $x_1 = x_{n+1} = 0$ and define

$$(Lx)_i = -\frac{1}{2}x_i v^* - \frac{1}{2}x_{i+1}u^*, \qquad 1 \le i \le n. \tag{14}$$

Finally, define $E : \mathcal{Y}_n \to \mathcal{Y}_n$ by

$$(Ex)_i = \frac{1}{2}(vx_iv^* + vx_{i+1}u^* + ux_{i-1}v^* + ux_iu^*), \qquad 2 \le i \le n. \tag{15}$$

Lemma 3.1. *One has*

$$TL = I_{\mathcal{Y}_n} - E.$$

Moreover, $\|L\| \le 1$.

Proof. Using (12), for $2 \le i \le n$ we have

$$[v, (Lx)_i] = -\frac{1}{2}vx_iv^* - \frac{1}{2}vx_{i+1}u^* + \frac{1}{2}x_i,$$
$$[u, (Lx)_{i-1}] = -\frac{1}{2}ux_{i-1}v^* - \frac{1}{2}ux_iu^* + \frac{1}{2}x_i.$$

Their sum is $x_i - (Ex)_i$. Also

$$\|(Lx)_i\| \le \frac{1}{2}\|x_i\| + \frac{1}{2}\|x_{i+1}\| \le \|x\|,$$

so $\|L\| \le 1$. □

The Cuntz relations give a unital $*$-isomorphism

$$\Theta : \mathcal{B} \longrightarrow M_2(\mathcal{B}), \qquad \Theta(z) = \begin{pmatrix} v^*zv & v^*zu \\ u^*zv & u^*zu \end{pmatrix}. \tag{16}$$

Indeed, the map

$$U : H \oplus H \longrightarrow H, \qquad U(\xi, \zeta) = v\xi + u\zeta,$$

is unitary by (12), and $\Theta(z) = U^*zU$. Its inverse is

$$\begin{pmatrix} z_{00} & z_{01} \\ z_{10} & z_{11} \end{pmatrix} \longmapsto vz_{00}v^* + vz_{01}u^* + uz_{10}v^* + uz_{11}u^*.$$

In particular,

$$\Theta((Ex)_i) = \frac{1}{2}\begin{pmatrix} x_i & x_{i+1} \\ x_{i-1} & x_i \end{pmatrix}. \tag{17}$$

Lemma 3.2 (Weighted spectral gap)**.** *Define*

$$\|x\|' = \sup_{2 \le i \le n} \left(2 - \frac{i^2}{n^2}\right)^{-1/2} \|x_i\|. \tag{18}$$

Then

$$\|Ex\|' \le \left(1 - \frac{1}{8n^2}\right)\|x\|'. \tag{19}$$

Consequently, $I - E$ is invertible and

$$\left\|(I-E)^{-1}\right\|_{\mathcal{Y}_n \to \mathcal{Y}_n} \le 8\sqrt{2}\, n^2. \tag{20}$$

Thus

$$R = L(I-E)^{-1} \tag{21}$$

is a right inverse for T and satisfies

$$\|R\| \le 8\sqrt{2}\, n^2. \tag{22}$$

Proof. By homogeneity, assume $\|x\|' \le 1$. Put $w_i = 2 - i^2/n^2$ for $1 \le i \le n+1$ and continue to use $x_1 = x_{n+1} = 0$. Notice that

$$w_{n+1} = 1 - \frac{2}{n} - \frac{1}{n^2} > 0 \qquad (n \ge 3),$$

so all these formal weights are positive. We have $\|x_i\|^2 \le w_i$ for $2 \le i \le n$, and the same inequality holds for $i = 1, n+1$ because the corresponding x_i is zero. Using the isometry (16), then bounding the norm of an operator matrix by the Frobenius norm of the scalar matrix of entry norms, gives

$$\begin{aligned}
\|(Ex)_i\|^2 &\le \frac{1}{4}\left(\|x_i\|^2 + \|x_{i+1}\|^2 + \|x_{i-1}\|^2 + \|x_i\|^2\right) \\
&\le \frac{1}{4}(w_i + w_{i+1} + w_{i-1} + w_i) \\
&= w_i - \frac{1}{2n^2} \\
&\le \left(1 - \frac{1}{4n^2}\right) w_i \\
&\le \left(1 - \frac{1}{8n^2}\right)^2 w_i.
\end{aligned}$$

This proves (19).

The weights $w_i^{-1/2}$ lie between $1/\sqrt{2}$ and 1, so

$$\|x\|' \le \|x\| \le \sqrt{2}\,\|x\|'. \tag{23}$$

The Neumann series in the primed norm therefore gives

$$\left\|(I-E)^{-1}x\right\| \le \sqrt{2}\sum_{m=0}^{\infty}\left(1 - \frac{1}{8n^2}\right)^m \|x\|' \le 8\sqrt{2}\, n^2\, \|x\|,$$

which is (20). Lemma 3.1 now yields $TR = I_{\mathcal{Y}_n}$ and (22). □

4 Point-source Green functions and killed walks

In this section, $\mathcal{Y}_n = \mathcal{B}^{n-1}$ carries the supremum norm, with coordinates indexed by $2, \dots, n$. For $j \in \{2, \dots, n\}$, let $\eta^{(j)} \in \mathcal{Y}_n$ be the scalar point source

$$\eta_i^{(j)} = \mathbf{1}_{\{i=j\}}\mathbf{1}, \qquad 2 \le i \le n. \tag{24}$$

The source used in Tao's nonlinear system is $\eta^{(n)}$.

Proposition 4.1 (Point-source Green-function bound)**.** *There is an absolute constant C_0 such that, for all $n \geq 3$ and all $j \in \{2, \dots, n\}$,*

$$\left\|(I-E)^{-1}\eta^{(j)}\right\| \leq C_0 n. \tag{25}$$

Consequently, if

$$a = n\eta^{(n)} = (0, \dots, 0, n\mathbf{1}) \in \mathcal{Y}_n, \tag{26}$$

then

$$\|Ra\| \leq C_0 n^2. \tag{27}$$

We first describe the iterated block coordinates. For $m \geq 1$ and a word $\alpha = (\alpha_1, \dots, \alpha_m) \in \{0,1\}^m$, put

$$s_0 = v, \qquad s_1 = u, \qquad s_\alpha = s_{\alpha_1} \cdots s_{\alpha_m}.$$

An induction on m, using (12), gives

$$s_\alpha^* s_\beta = \mathbf{1}_{\{\alpha=\beta\}}\mathbf{1}, \qquad \sum_{\alpha \in \{0,1\}^m} s_\alpha s_\alpha^* = \mathbf{1}. \tag{28}$$

Thus

$$U_m : H^{\oplus 2^m} \longrightarrow H, \qquad U_m((\xi_\alpha)_\alpha) = \sum_\alpha s_\alpha \xi_\alpha,$$

is unitary, and

$$\Theta_m : \mathcal{B} \longrightarrow M_{2^m}(\mathcal{B}), \qquad \Theta_m(z) = (s_\alpha^* z s_\beta)_{\alpha,\beta \in \{0,1\}^m} = U_m^* z U_m \tag{29}$$

is a unital isometric $*$-isomorphism.

For $\alpha, \beta \in \{0,1\}^m$, define the difference walk

$$\sigma_k(\alpha,\beta) = \sum_{r=1}^{k} (\beta_r - \alpha_r), \qquad 1 \leq k \leq m. \tag{30}$$

Lemma 4.2 (Path formula)**.** *Let $m \geq 1$. For every $x \in \mathcal{Y}_n$, every $2 \leq i \leq n$, and every $\alpha, \beta \in \{0,1\}^m$,*

$$\left[\Theta_m((E^m x)_i)\right]_{\alpha,\beta} = 2^{-m} \left(\prod_{k=1}^{m} \mathbf{1}_{\{2 \leq i+\sigma_k(\alpha,\beta) \leq n\}} \right) x_{i+\sigma_m(\alpha,\beta)}. \tag{31}$$

Proof. For $m = 1$, formula (31) is exactly (17), with the convention $x_1 = x_{n+1} = 0$. Assume the formula at level m. We use the prefix concatenation convention

$$\widetilde{\alpha} = (a, \alpha), \qquad \widetilde{\beta} = (b, \beta), \qquad a, b \in \{0,1\},$$

so that $s_{\widetilde{\alpha}} = s_a s_\alpha$. Then

$$\sigma_1(\widetilde{\alpha}, \widetilde{\beta}) = b - a, \qquad \sigma_{k+1}(\widetilde{\alpha}, \widetilde{\beta}) = b - a + \sigma_k(\alpha,\beta) \quad (1 \leq k \leq m). \tag{32}$$

Moreover,

$$\left[\Theta_{m+1}(z)\right]_{\widetilde{\alpha},\widetilde{\beta}} = \left[\Theta_m(s_a^* z s_b)\right]_{\alpha,\beta}.$$

Apply this with $z = (E^{m+1}x)_i = E(E^m x)_i$. By (17), the right-hand side equals

$$\frac{1}{2}\left[\Theta_m((E^m x)_{i+b-a})\right]_{\alpha,\beta}$$

when $2 \le i+b-a \le n$, and equals zero otherwise. The induction hypothesis at the shifted site $i+b-a$, together with (32), gives (31) at level $m+1$. □

Remark 4.3 (Killed-walk interpretation)**.** *For fixed i, α, β, the sequence*

$$i + \sigma_1(\alpha,\beta), \dots, i + \sigma_m(\alpha,\beta)$$

is a walk with increments in $\{-1, 0, 1\}$. The product of indicators in (31) *kills the coefficient as soon as this difference walk leaves $\{2, \dots, n\}$. For a general operator-valued x, the terminal coefficient is the operator $x_{i+\sigma_m}$. For a scalar point source $\eta^{(j)}$, it is either* 0 *or* $\mathbf{1}$*. This is precisely why the operator-valued matrix becomes a scalar matrix tensored with $\mathbf{1}_H$ for the distinguished sources considered below.*

For $d \in \mathbb{Z}$, define a scalar matrix on $\ell_2(\{0,1\}^m)$ by

$$K_{m,d}(\alpha,\beta) = 2^{-m}\mathbf{1}_{\{|\beta|-|\alpha|=d\}}, \tag{33}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Lemma 4.4 (Binomial block bound)**.** *There is an absolute constant C_1 such that*

$$\|K_{m,d}\|_{\ell_2\to\ell_2} \le \frac{C_1}{\sqrt{m+1}} \tag{34}$$

for every $m \ge 0$ and every integer d.

Proof. Let

$$\Omega_p = \{\alpha \in \{0,1\}^m : |\alpha| = p\}.$$

With respect to the orthogonal decompositions by Hamming weight, $K_{m,d}$ is the orthogonal direct sum, over all p for which $0 \le p, p+d \le m$, of the constant rectangular matrices

$$2^{-m} J_{\binom{m}{p},\binom{m}{p+d}}.$$

The domain and range subspaces belonging to distinct p are mutually orthogonal. Since $\|J_{r,s}\| = \sqrt{rs}$,

$$\begin{aligned}\|K_{m,d}\| &= 2^{-m}\max_p \sqrt{\binom{m}{p}\binom{m}{p+d}} \\ &\le 2^{-m}\max_{0\le q\le m}\binom{m}{q}.\end{aligned}$$

The central-binomial estimate

$$\max_q \binom{m}{q} \le C_1 \frac{2^m}{\sqrt{m+1}}$$

proves the claim. □

Lemma 4.5 (Diffusive point-source decay)**.** *There are absolute constants $C_2, c_2 > 0$ such that, for all $n \geq 3$, $j \in \{2, \ldots, n\}$, and $m \geq 0$,*

$$\left\| E^m \eta^{(j)} \right\| \leq \frac{C_2}{\sqrt{m+1}} \exp\left(-c_2 \frac{m}{n^2}\right). \tag{35}$$

Proof. The case $m = 0$ is immediate after enlarging C_2, so assume $m \geq 1$. Fix $2 \leq i \leq n$. Lemma 4.2 gives the explicit identity

$$\Theta_m((E^m \eta^{(j)})_i) = A^{(j)}_{m,i} \otimes \mathbf{1}_H, \tag{36}$$

where the scalar matrix $A^{(j)}_{m,i}$ is given by

$$A^{(j)}_{m,i}(\alpha, \beta) = 2^{-m} \left(\prod_{k=1}^{m} \mathbf{1}_{\{2 \leq i + \sigma_k(\alpha,\beta) \leq n\}} \right) \mathbf{1}_{\{i + \sigma_m(\alpha,\beta) = j\}}. \tag{37}$$

Thus every entry is either 0 or 2^{-m}. A nonzero entry requires

$$|\beta| - |\alpha| = \sigma_m(\alpha, \beta) = j - i.$$

The killing conditions only delete entries, and therefore

$$0 \leq A^{(j)}_{m,i} \leq K_{m,j-i} \tag{38}$$

entrywise.

For nonnegative scalar matrices, entrywise domination implies domination of the ℓ_2 operator norm. Indeed, for every complex vector z,

$$\left| A^{(j)}_{m,i} z \right| \leq A^{(j)}_{m,i} \, |z| \leq K_{m,j-i} \, |z|$$

coordinatewise. Hence

$$\left\| A^{(j)}_{m,i} z \right\|_2 \leq \| K_{m,j-i} \| \, \|z\|_2 \, .$$

Furthermore,

$$\left\| A^{(j)}_{m,i} \otimes \mathbf{1}_H \right\| = \left\| A^{(j)}_{m,i} \right\|_{\ell_2 \to \ell_2} . \tag{39}$$

Since Θ_m is isometric, Lemma 4.4 gives

$$\left\| E^m \eta^{(j)} \right\| \leq \frac{C_1}{\sqrt{m+1}}. \tag{40}$$

We combine this diffusive estimate with Tao's spectral gap. Put $q_n = 1 - 1/(8n^2)$. Lemma 3.2 and (23) imply

$$\|E^r y\| \leq \sqrt{2}\, q_n^r \, \|y\| \qquad (r \geq 0, \ y \in \mathcal{Y}_n). \tag{41}$$

Given $m \geq 1$, write $m = r + s$ with $r = \lfloor m/2 \rfloor$. Then

$$\begin{aligned} \left\| E^m \eta^{(j)} \right\| &\leq \sqrt{2}\, q_n^r \left\| E^s \eta^{(j)} \right\| \\ &\leq \frac{\sqrt{2} C_1 q_n^r}{\sqrt{s+1}} \\ &\leq \frac{C_2}{\sqrt{m+1}} \exp\left(-c_2 \frac{m}{n^2}\right), \end{aligned}$$

where we used $1 - t \leq e^{-t}$, $s + 1 \geq (m+1)/2$, and $r \geq (m-2)/2$, absorbing the finitely many small values of m into C_2. ∎

Proof of Proposition 4.1. The Neumann series converges in norm by Lemma 3.2. Uniformly in j,

$$\begin{aligned}\left\|(I-E)^{-1}\eta^{(j)}\right\| &\le \sum_{m=0}^{\infty}\left\|E^m\eta^{(j)}\right\| \\ &\le C_2\sum_{m=0}^{\infty}\frac{e^{-c_2 m/n^2}}{\sqrt{m+1}} \\ &\le C_0 n.\end{aligned}$$

For the last inequality, compare the sum with

$$1+\int_0^{\infty}(t+1)^{-1/2}e^{-c_2 t/n^2}\,dt = O(n).$$

Finally, $R = L(I-E)^{-1}$, $\|L\|\le 1$, and $a = n\eta^{(n)}$, so

$$\|Ra\| \le n\left\|(I-E)^{-1}\eta^{(n)}\right\| \le C_0 n^2.$$

□

Remark 4.6. *Proposition 4.1 is a point-to-bulk resolvent estimate for the killed block walk encoded by E. Its scalar reduction relies on the source being a scalar multiple of the identity at one coordinate. For a general operator-valued source, the terminal coefficient in* (31) *remains operator-valued, and the scalar binomial comparison used above does not directly yield the same estimate.*

5 Solving the nonlinear system at the fourth-power scale

For $b = (b_1,\dots,b_n) \in \mathcal{X}_n$, define $F : \mathcal{X}_n \to \mathcal{Y}_n$ and $G : \mathcal{X}_n \times \mathcal{X}_n \to \mathcal{Y}_n$ by

$$(Fb)_i = \begin{cases} -ib_{i+1}, & 2\le i\le n-1, \\ 0, & i = n, \end{cases} \tag{42}$$

$$G(b,c)_i = -b_i[u,c_n], \qquad 2\le i\le n. \tag{43}$$

Then

$$\|F\|\le n-1, \qquad \|G\|\le 2. \tag{44}$$

The equations (7)–(8) are exactly

$$Tb = a + \delta F(b) + \delta G(b,b), \tag{45}$$

with a as in (26).

Remark 5.1. *Formula* (42) *is obtained directly from* (7)*: the coordinate indexed by i contains $-ib_{i+1}$. This indexing is the one used throughout the proof. Only the bound $\|F\|\le n-1$ is relevant to the estimates.*

Proposition 5.2. *There are absolute constants $c_0, C_3 > 0$ such that, for every $n \geq 3$ and*

$$0 < \delta \leq c_0 n^{-4},$$

there exists $b = (b_1, \ldots, b_n) \in \mathcal{X}_n$ solving (45) and satisfying

$$\max_{1 \leq i \leq n} \|b_i\| \leq C_3 n^2. \tag{46}$$

Proof. Let C_0 be as in Proposition 4.1, and set

$$M = C_0 n^2.$$

Then $\|Ra\| \leq M$. Consider the closed ball

$$\mathcal{C} = \{b \in \mathcal{X}_n : \|b\| \leq 2M\}$$

and the map

$$\Psi(b) = R\big(a + \delta F(b) + \delta G(b, b)\big). \tag{47}$$

A fixed point of Ψ solves (45), since $TR = I$.

Let $C_R = 8\sqrt{2}$, so that $\|R\| \leq C_R n^2$. If $b \in \mathcal{C}$, then, by (44),

$$\begin{aligned} \|\Psi(b)\| &\leq M + \delta \|R\| \left(\|F\| \|b\| + \|G\| \|b\|^2\right) \\ &\leq M + \delta C_R n^2 \left(2M(n-1) + 8M^2\right). \end{aligned}$$

Thus $\Psi(\mathcal{C}) \subseteq \mathcal{C}$ provided

$$\delta C_R n^2 \left(2(n-1) + 8M\right) \leq 1. \tag{48}$$

For $b, c \in \mathcal{C}$, bilinearity gives

$$G(b, b) - G(c, c) = G(b - c, b) + G(c, b - c),$$

and hence

$$\begin{aligned} \|\Psi(b) - \Psi(c)\| &\leq \delta \|R\| \left(\|F\| + \|G\| (\|b\| + \|c\|)\right) \|b - c\| \\ &\leq \delta C_R n^2 \left((n-1) + 8M\right) \|b - c\|. \end{aligned}$$

Choose explicitly

$$c_0 = \min\left\{1, \frac{1}{4C_R(8C_0 + 2/3)}\right\}. \tag{49}$$

If $0 < \delta \leq c_0 n^{-4}$ and $n \geq 3$, then

$$\begin{aligned} \delta C_R n^2 \left(2(n-1) + 8M\right) &\leq c_0 C_R \left(\frac{2(n-1)}{n^2} + 8C_0\right) \\ &\leq c_0 C_R (2/3 + 8C_0) \leq \frac{1}{4}, \end{aligned}$$

so $\Psi(\mathcal{C}) \subseteq \mathcal{C}$. Likewise, its Lipschitz coefficient is at most

$$\begin{aligned} \delta C_R n^2 \left((n-1) + 8M\right) &\leq c_0 C_R \left(\frac{n-1}{n^2} + 8C_0\right) \\ &\leq c_0 C_R (2/3 + 8C_0) \leq \frac{1}{4}. \end{aligned}$$

Thus Ψ is a strict contraction of $\mathcal{C}$. The contraction mapping theorem gives a fixed point $b \in \mathcal{C}$, which satisfies (46) with $C_3 = 2C_0$. $\square$

6 Proof of the main theorem

Proof of Theorem 1.1. Fix $n \geq 3$ and set

$$\delta = c_0 n^{-4},$$

where c_0 is as in Proposition 5.2. Let $b_1, \dots, b_n$ be the solution furnished there. Then (7) and (8) hold, and

$$B_n := \max_i \|b_i\| \leq C_3 n^2.$$

Apply Corollary 2.3 with $\mu = 1/2$ and with the Cuntz isometries u, v, for which $\|u\| = \|v\| = 1$. The geometric sums satisfy

$$\sum_{i=1}^{n} \mu^{n-i-1} \leq \frac{1}{\mu(1-\mu)} = 4,$$
$$\sum_{i=1}^{n} \mu^{n-i+1} \leq \frac{\mu}{1-\mu} = 1.$$

It follows from (9) and (10) that

$$\left\| D_{1/2} \right\| \leq 6\delta^{-1} + n + 4B_n \leq C_4 n^4, \tag{50}$$
$$\left\| X_{1/2} \right\| \leq 1 + \delta B_n \leq C_5. \tag{51}$$

Furthermore,

$$\begin{aligned} &\|[v, b_1] + \delta b_2 + \delta b_1 [u, b_n]\| \\ &\qquad \leq 2B_n + \delta B_n + 2\delta B_n^2 \leq C_6 n^2, \end{aligned}$$

where we used $\delta = O(n^{-4})$ and $B_n = O(n^2)$. Hence

$$\left\| [D_{1/2}, X_{1/2}] - \mathbf{1}_{M_n(\mathcal{B})} \right\| \leq C_6 n^2 2^{-(n-1)}. \tag{52}$$

We now choose n and keep track of the ceiling. Put

$$L = \log \frac{1}{\varepsilon} \geq \log 2.$$

Choose a constant $B > 0$, to be fixed below, and set

$$n = \max\{3, \lceil BL \rceil\}. \tag{53}$$

Then

$$BL \leq n \leq A_B L, \qquad A_B = \max\left\{ B + \frac{1}{\log 2}, \frac{3}{\log 2} \right\}. \tag{54}$$

Consequently,

$$\begin{aligned} C_6 n^2 2^{-(n-1)} &\leq 2C_6 A_B^2 L^2 e^{-B(\log 2)L} \\ &= \left(2C_6 A_B^2 L^2 e^{-(B\log 2 - 1)L} \right) e^{-L}. \end{aligned}$$

For B sufficiently large, the function $L \mapsto L^2 e^{-(B\log 2-1)L}$ is decreasing on $[\log 2, \infty)$, and

$$2C_6 A_B^2 (\log 2)^2 e^{-(B\log 2-1)\log 2} \le 1.$$

Such a B exists because the last exponential decays exponentially in B, whereas A_B^2 grows only quadratically. Fixing one such B, we obtain

$$C_6 n^2 2^{-(n-1)} \le e^{-L} = \varepsilon \qquad \text{and} \qquad n \le A_B L. \tag{55}$$

For this choice of n, (50)–(52) give

$$\left\| [D_{1/2}, X_{1/2}] - \mathbf{1} \right\| \le \varepsilon \qquad \text{and} \qquad \left\| D_{1/2} \right\| \left\| X_{1/2} \right\| \le Cn^4 \le CA_B^4 \left(\log \frac{1}{\varepsilon} \right)^4 .$$

Finally, $M_n(B(H)) \cong B(H^{\oplus n}) \cong B(H)$ isometrically because H is infinite-dimensional. Transporting $D_{1/2}, X_{1/2}$ through such a unitary identification proves the theorem. □

Remark 6.1 (What the argument does and does not optimize)**.** *The global right-inverse bound remains $\|R\| = O(n^2)$. The improvement comes from replacing the worst-case estimate $\|R\| \|a\| = O(n^3)$ by the point-source estimate $\|Ra\| = O(n^2)$. In the fixed-point inequalities, the dominant quantity is then*

$$\|R\| \left(\|F\| + \|Ra\| \right) = O(n^4).$$

Thus the present estimates naturally stop at $\delta^{-1} = O(n^4)$. This is not a proof that exponent four is a barrier for Tao's parametrization. A smaller exponent could follow from a sharper bound for R on the range of F and G, from cancellation in the nonlinear correction, or from a modified commutator template in which the quadratic term is smaller or more localized.

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